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COMMENT

Critical exponent of a directed self-avoiding walk

H W J Blöte and H J Hilhorst

Laboratorium voor Technische Natuurkunde, Lorentzweg 1, 2628 CJ Delft, The Netherlands

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Abstract. We have investigated numerically and analytically the directed self-avoiding walk problem recently proposed and studied by Chakrabarti and Manna. We find a different result for the exponent ν, namely $\nu = 1$.

Recently, a directed self-avoiding random walk problem on the simple quadratic lattice was defined and investigated by Chakrabarti and Manna (1983). In this problem, each step has a length of one lattice unit, but the direction of the step is subject to a restriction: steps may occur in the $\pm x$ and the $+y$ directions; steps in the $-y$ direction are forbidden. All allowed self-avoiding walks have equal weight. The average end-to-end distance for walks of N steps is denoted as $\bar{R}_N$. Chakrabarti and Manna (CM) investigated the behaviour of $\bar{R}_N$ as a function of N in the limit $N \to \infty$. To this purpose, they numerically calculated $\bar{R}_N$ for N values up to 14 by means of a computer. They found that, for large N,

$$\bar{R}_N \sim N^{\nu} \tag{1}$$

with $\nu = 0.86 \pm 0.02$. This value was estimated from a comparison between $\log \bar{R}_N$ and $\log N$.

We have calculated estimates ν_N of the exponent ν, defined as

$$\nu_N = \log[\bar{R}_N/\bar{R}_{N-1}]/\log[N/(N-1)] \tag{2}$$

from the $\bar{R}_N$ data given by CM. The behaviour of ν_N thus obtained suggests the possible presence of rounding errors in the CM data, in particular for the higher N values. For this reason we have computed some results for $\bar{R}_N$ using a machine accuracy of 16 decimal places (table 1).

The results for $N \geq 8$ show increasing differences from those of CM. Estimates ν_N from our results are also shown in table 1. For higher values of N, they increase significantly above the estimate of CM. We have also shown increments $\bar{R}_N - \bar{R}_{N-1}$ in table 1: these numbers rapidly approach a constant value $\frac{1}{2}$. This corresponds to linear behaviour of $\bar{R}_N$ for high N, hence to $\nu = 1$. The difference with $\nu = 0.86$ as estimated by CM can be explained by the unfortunate approximate cancellation of two effects: the aforementioned rounding errors, and nonlinear behaviour of $\log \bar{R}_N$ as a function of $\log N$ for N between 10 and 14.

To check our numbers we did an analytic calculation. Let $G_N(x, y)$ be the number of N-step walks between the origin and (x, y). Then

$$\bar{R}_N = \sum_{x=-\infty}^{\infty} \sum_{y=0}^{\infty} (x^2+y^2)^{1/2} G_N(x, y) \Big/ \sum_{x=-\infty}^{\infty} \sum_{y=0}^{\infty} G_N(x, y). \tag{3}$$

Table 1. Numerical results for the average lengths $\bar{R}_N$ of walks of N steps, estimates of the exponent ν, and the increments of $\bar{R}_N$.

N	$\bar{R}_N$	ν_N	$\bar{R}_N - \bar{R}_{N-1}$
1	1.000 00	—	1.000 0
2	1.665 26	0.735 8	0.665 3
3	2.225 46	0.715 2	0.560 2
4	2.785 23	0.779 9	0.559 8
5	3.322 00	0.789 8	0.536 8
6	3.850 99	0.810 5	0.529 0
7	4.372 81	0.824 4	0.521 8
8	4.890 44	0.837 8	0.517 6
9	5.404 80	0.849 1	0.514 4
10	5.916 85	0.859 1	0.512 0
11	6.427 08	0.867 9	0.510 2
12	6.935 91	0.875 7	0.508 8
13	7.443 61	0.882 6	0.507 7
14	7.950 39	0.888 8	0.506 8
15	8.456 40	0.894 3	0.506 0
16	8.961 79	0.899 4	0.505 4
17	9.466 63	0.904 0	0.504 8
18	9.971 01	0.908 2	0.504 4
19	10.475 00	0.912 0	0.504 0
20	10.978 64	0.915 5	0.503 6
21	11.481 98	0.918 8	0.503 3
22	11.985 06	0.921 8	0.503 1
23	12.487 90	0.924 6	0.502 8
24	12.990 54	0.927 2	0.502 6

To find $G_N(x, y)$ we have to do some combinatorics. We define a $+x$ segment ($-x$ segment) of a walk as a sequence of one or more consecutive steps in the $+x$ direction ($-x$ direction) preceded and followed by a step in the y direction. Let the number of $+x$ segments in a walk be t_+ and the number of $-x$ segments t_-. Each of these segments has a definite y coordinate which, for a walk between $(0, 0)$ and (x, y), may take any of the $y+1$ integer values from 0 to y. The number of ways of arranging the $+x$ segments and the $-x$ segments with respect to their y coordinate is therefore $(y+1)!/[t_+!t_-!(y+1-t_+-t_-)!]$. How many walks are there with a given arrangement? The walk has N steps of which y are in the y direction, and hence $N-y$ in the $\pm x$ direction. Since the net displacement along the x axis is x, there must be $\frac{1}{2}(N+x-y)$ steps in the $+x$ direction (i.e. contained in the $+x$ segments) and $\frac{1}{2}(N-x-y)$ steps in the $-x$ direction (i.e. contained in the $-x$ segments). Hence the total number $m(t_+, t_-)$ of walks with given t_+ and t_- is the previous combinatorial expression multiplied by the number of ways of distributing $\frac{1}{2}(N+x-y)$ steps over t_+ segments and $\frac{1}{2}(N-x-y)$ step over t_- segments, i.e.

$$m(t_+, t_-) = \frac{(y+1)!}{t_+!\,t_-!\,(y+1-t_+-t_-)!} \binom{\frac{1}{2}(N+x-y)-1}{t_+-1} \binom{\frac{1}{2}(N-x-y)-1}{t_--1}. \tag{4}$$

Finally

$$G_N(x, y) = \sum_{t_+=0}^{y+1} \sum_{t_-=0}^{y+1-t_+} m(t_+, t_-). \tag{5}$$

For large N, we expect t_+, t_- and y also to become large and we may use Stirling's formula in each of the factors in (4). The summations in (5) may then be replaced with integrations, and we can obtain, using the steepest-descent method, an asymptotic expansion of the integrals. A maximum of the integrand occurs when, to leading order in N,

$$t_{\pm}=[\tfrac{1}{2}\eta/(n\pm\xi)][1\pm\xi-(1-\xi^2-2\eta+2\eta^2)^{1/2}]N, \tag{6}$$

where $\xi=x/N$ and $\eta=y/N$. The expression (3) may be evaluated subsequently by the same technique. $G_N(x, y)$ appears to have significant values only close to the point $x=0$, $y=\frac{1}{2}N$, and thus we find the leading term in the expansion of $\bar{R}_N$:

$$\bar{R}_N\simeq\tfrac{1}{2}N. \tag{7}$$

This shows that

$$\nu=1 \tag{8}$$

and confirms the prefactor $\frac{1}{2}$ strongly suggested by the numerical results of table 1.

Reference

Chakrabarti B K and Manna S S 1983 *J. Phys. A: Math. Gen.* **16** 1983 L113